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Teacher Candidates' Understanding and Appraoches to Errors About Matrices

Şükrü İLGÜN[1]
Kafkas University

Solmaz Damla GEDİK ALTUN[2]
Nevşehir Hacı Bektaş Veli University

Alper Cihan KONYALIOĞLU[3]
Atatürk University

Abstract

The aim of this study is to examine the ability of pre-service mathematics teachers to detect errors made in solving questions about matrices. The study particularly focused on revealing the internalization of the teachings such as the meanings and relational dimensions of concepts and operations about matrix. The study was conducted with 26 teacher candidates at a university in the Eastern Anatolia Region. They were given a written exam, and their responses were analyzed by two field experts. The results showed that the pre-service teachers did not fully understand the concepts and operations of matrices. They made a variety of errors, including misconceptions and incomplete understanding. They were also not very good at solving proof-based questions. However, they were more successful at solving problems that were based on plain logic or could be solved using rules.

Keywords: Error, Error approximation, Linear Algebra, Matrices

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[1]Assoc. Prof., Faculty of Education, Kafkas University, Kars , Türkiye, ORCID: 0000- 0002-2842-2032

Correspondence: mat.ilgun@hotmail.com

[2] Assist. Prof., Faculty of Education, Nevşehir Hacı Bektaş Veli University, Nevşehir, Türkiye, ORCID: 0000-0002-6205-6603 Email: sdgedik@nevsehir.edu.tr

[3] Prof., Faculty of Education, Atatürk University, Erzurum, Türkiye, ORCID: 0000-0002-6009-4251 Email: ackonyali@atauni.edu.tr

Introduction

It is important to know the errors that students make and the misconceptions they experience in learning environments so that meaningful learning can take place and a qualified education can be provided (Altıntaş et al., 2021). According to Baştürk (2014), it would be possible to say that error has an indisputable place in mathematics. In this context, with some changes made in mathematics education, mathematical errors have gained an important dimension.

An important way to use error in mathematics in a positive way is error detection studies. With these studies, it can be revealed whether the correct information is truly internalized or not. Is the indicator of knowing just giving the right answer, making the right solution? It is of course important to give the right answer and to make the right solution, but the individuals who are assumed to know about any subject should have the ability to see the errors made about what they knows. It should be known that this is an indication that knowing what is true is necessary but not sufficient for learning (Durkaya et al. 2011).

Being able to detect errors correctly is a very important phenomenon. While Tirosh (2000) explains the fact that the subject knowledge reach a certain level explaining with awareness of errors, this situation is also seen as a factor of having in-depth subject knowledge emphasized in NCTM (2000). Again, error detection, Ball et al. (2008), is included in the special content knowledge that a person who can correctly question the error has internalized the related concept to a large extent (Konyalıoğlu et al., 2010; Konyalıoğlu et al., 2012). The correct approach to the error and the correct solution proposal is one of the components that can be used in determining the adequacy of the subject knowledge (Durkaya et al. 2011). Borasi (1986; 1989) stated that the nature of mathematics can be better understood with error approaches (Demirci et al., 2017).

Linear algebra is one of the branches of mathematics related to abstract structures that represent various concepts and systems that contain different properties by nature (Mutuk, 2018). Linear algebra is treated as vector and matrix algebra. Students encounter many new concepts and definitions in this branch. Studies (eg Britton & Henderson, 2009) reveal that students have various difficulties especially in learning and making sense of these concepts and definitions. In discussions about teaching and learning linear algebra (Dorier & Sierpinska, 2001), It is claimed that linear algebra courses are poorly designed and poorly administered and linear algebra continues to be a difficult subject cognitively and conceptually, no matter how it is taught. Haddad (1999) devided the learning difficulties encountered in linear algebra into 3 categories as; the nature, teaching and learning of linear algebra; difficulties faced by students based on their inability to think abstractly; and the basic axioms of the subject and the lack of foundations of mathematics. According to Harel (1989a), the reason for the learning difficulties of basic notations is the attempt to build the foundations of mathematics with an abstract structure on a weak conceptual foundation. On the other hand, students operate without a conceptual understanding (Harel, 1989b). Carlson (1993) states that students are generally successful in tasks involving simple computational algorithms (matrix multiplication and systems of simple linear equations), and they make errors in tasks related to linear independence and transformations.

Despite attempts to adapt the curriculum to students' interests and learning processes, it is important to acknowledge the fact that linear algebra has been and will continue to be a difficult subject for most students. Two types of sources of difficulties experienced by students are; conceptual difficulties arising from the nature of linear algebra and the type of thinking required to understand linear algebra, namely cognitive difficulties. However, it should be understood that these two aspects are often inseparable in actual learning and knowing processes (Dorier & Sierpinska, 2001).

Since linear algebra has a cumulative structure, the previous concept learned has an important place in the next concept teaching. If meaningful transitions between concepts cannot be achieved, permanent learning cannot take place. In this context, it is of great importance for teachers to gain experience in this regard. For this, they need to plan their lessons by seeing the errors that students have as a result of their own teachings and that other students have (Altıntaş et al., 2020). For this purpose, it is of great importance for teachers to make their plans accordingly by receiving feedback from students through error-based activities. According to Altıntaş (2020), from this aspect, teachers can see and use errors as a learning tool. It can be thought that linear algebra lessons taught using error-based activities can make a significant contribution to students' motivation, development and creativity and can improve their mathematical thinking and meaningful learning skills. Accordingly, in this study, mathematics teacher candidates were given incorrect solutions of linear equations and the students were asked to find the errors in these solutions. The aim of this research is to determine how pre-service mathematics teachers understand the matrix and what kind of errors they make in the matrix. The problem statement of the research is determined as “How do pre-service mathematics teachers understand the matrix and what kind of errors do they make?”.

Method

Research Model

In the study, pre-service teachers were asked questions that should use the commutative property, which is one of the key words in multiplication in matrices, and questions that were solved incorrectly and that the error should be found and explained. It is aimed to reach in-depth information by asking to reveal the existing errors in the questions whose solutions are presented. In this respect, case study is based on qualitative research approaches (Davey, 2009; Büyüköztürk, 2018). In other words, a case study is a model in which individuals, events or processes are handled comprehensively and allows to reveal how various factors affect the situation under consideration (Yıldırım & Şimşek, 2018).

Research Group

The participant group of the study consists of 26 students attending a mathematics teaching program at a university in Türkiye. Students were coded as K1, K2, K3,… within the scope of the study.

Data Collection Tools

Since the study was handled within the scope of the linear algebra course, all the students who took the course participated in the course. As a data collection tool, interviews were held with two experts and four open-ended questions and their solutions were prepared in order to better determine the learning of the students, which were processed in the matrices and thought to have a positive effect on the results of the research to be conducted. The prepared measurement tool was applied in an undergraduate course midterm exam. While preparing the questions, especially the literature was examined and the perspectives of mathematics education experts were taken into account. The scope of the questions asked is in general about the features of the multiplication operation in matrices and is indicated in Table 1.

Table 1. Contents of prepared questions

Question	Content of the Question
Question 1	Importance of starting from right or left (Swap feature)
Question 2	Using the union feature as a commutative feature
Question 3	Whether exponentiation is related to inversion and transposition
Question 4	Zero divisor, unit element feature

Two stages were taken into consideration while analyzing the study. In the first stage, the answers given by the students as correct, partially correct, incorrect, no answer were categorized by focusing on the answers produced by the students. From this category; 'Correct: The answer given contains all academic components, Partially Correct: The answer does not include all academic components, Incorrect: The answer does not include academic truths, contains errors, misconceptions and wrongs, No answer: The answer is absent or left blank.'

In order to explain the questions in depth, analysis was carried out in the second stage. At this stage, considering the answers to the questions, codes and categories related to errors, misconceptions, and errors were determined. The code and category process of the research was carried out independently by two researchers. The obtained analyzes were presented in tables and general comments were made. Images were created by quoting the answers produced by the students.

Findings

In the first part of this section, an overview of the answers of the students was made, and in the second part, the errors and misconceptions that the students revealed while analyzing the questions were discussed.

Pre-service Teachers' Overview of the Questions

In the first analysis stage, the results of the pre-service teachers' answers to the questions asked, in which the answers were given incorrectly or correctly, the proof of which was presented, and whether the feature was true or false, are shown in Table 2.

Table 2. Pre-service teachers' answers to the questions

	Correct		Partially Correct		Incorrect		No Answer	
Questions	F	%	F	%	f	%	f	%
1	11	%43	1	%4	14	%53	0	%0
2	0	%0	0	%0	26	%100	0	%0
3	4	%15	10	%38	11	%43	1	%4
4	3	%11	1	%4	22	%85	0	%0

Looking at the results given in Table 2, it is seen that the question with the most correct answers is the first question, the question with the most wrong answers and even no correct and partially correct answers is the second question, and the only question in which the answer is never produced by certain pre-service teachers is the third question.

Discovery of Pre-service Teachers' Errors and Misconceptions in Questions

In the second stage analyzes, the phenomenon of finding errors in the solutions of pre-service teachers, discovering misconceptions and identifying wrong answers were discussed. For this, the answers of the teacher candidates were examined as a whole by the researchers. Afterwards, the types of answers that teacher candidates frequently repeated while producing their answers were determined. It is seen that these response types are especially shaped as follows; *'Doing a validation study, making a proof, directly using the feature, explaining with an example-numerical example, checking by doing the same solution, finding the error by comparing, trying to persuade'.* These answers produced by the pre-service teachers were determined as codes. It is understood that pre-service teachers usually give answers as 'correct, partially correct, incorrect' while answering the questions and then try to prove these 'correct, partially correct, incorrect' answers. The researchers evaluated these responses within the framework of the general view and addressed them as a category.

Table 3. The answers of the pre-service teachers about the question 1.

Category	Code	f	%
Correct Answer	Making proof	K5, K16, K24, K25	%16
	Validation study	K1, K9, K12, K20, K21	%16
	Directly using the feature	K15, K18	%8
Partially Correct Answer	Validation work	K23	%4
Incorrect Answer	Validation work	K4, K6, K8, K10, K11,K14, K19, K22	%32
	inability to find error	K13, K17, K26	%12
	Looking for the error in the wrong place	K2, K3	%8
	Searching for error by comparison	K7	%4

Looking at Table 3, it is seen that eleven of the pre-service teachers gave correct answers to the question, one partially correct answer and sixteen incorrect answers. In order to reveal the error in the given solution, the pre-service teachers used the codes of making proof, validation work, and using the feature directly while producing the correct answer; they do validation work while producing partially correct answers; While producing the wrong answer, it is seen that they use the codes of verification, finding the error, searching the error in the wrong place, finding the error by comparing. Below are the visuals and comments of some pre-service teachers' answers to question 1.

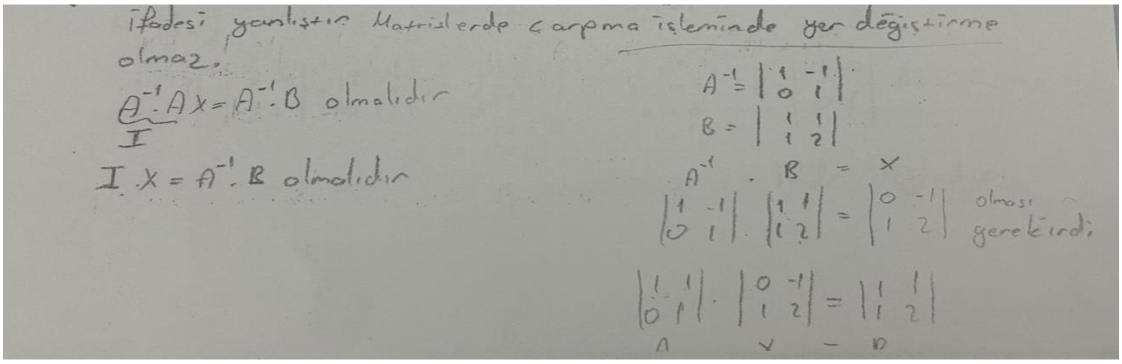


Figure 1. Example of correct answer given by the pre-service teacher to the question 1

Figure 1 shows the correct answer given by the K5 pre-service teacher for the 1st question. The pre-service teacher correctly expressed the answer to the question. While producing the correct answer, he told how to multiply the matrices and also stated how the correct operation should be.

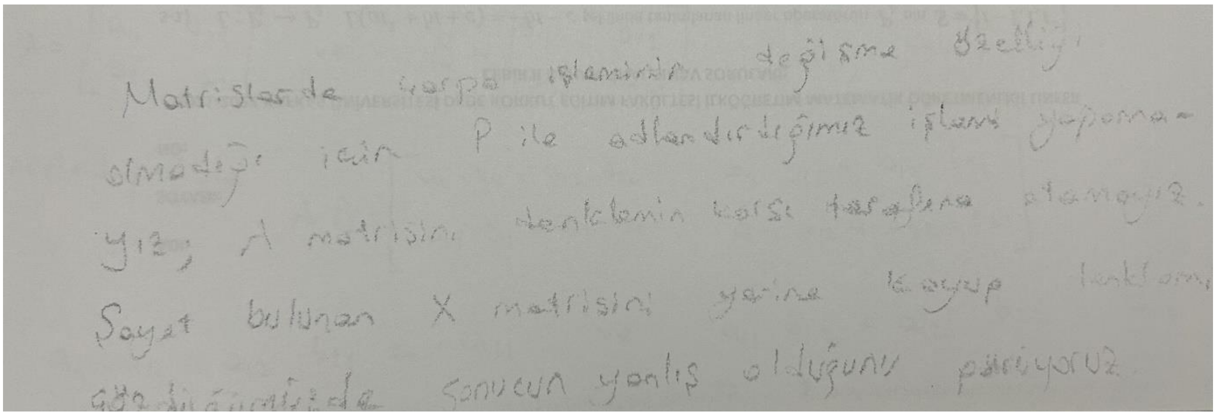


Figure 2. Example of partially correct answer given by the pre-service teacher to question 1.

Figure 2 shows the partially correct answer given by the K23 pre-service teacher for the 1st question. The pre-service teacher said that the existence of commutative feature is important when multiplying in matrices and if this exists, multiplication will be done in matrices. However, the pre-service teacher could not realize how the A matrix would be moved across and could not interpret the contribution of the inverse of the A matrix to the process. Therefore, he could not quite catch the right part of the process.

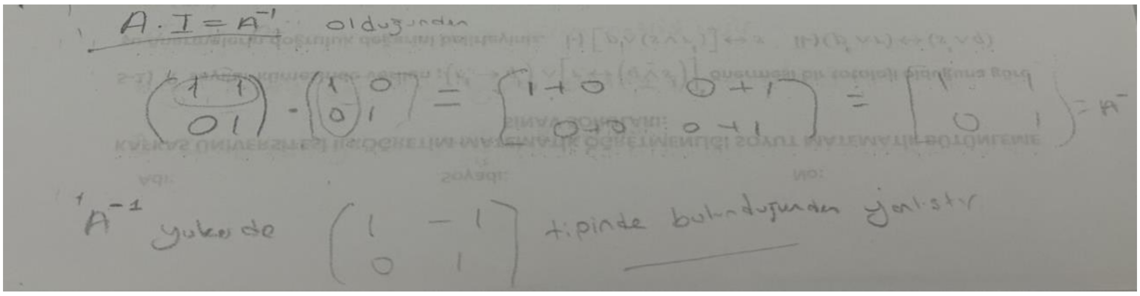


Figure 3. Example of wrong answer given by pre-service teacher to the question 1.

In Figure 3, the wrong answer given by the K2 pre-service teacher for the 1st question is seen. The pre-service teacher found the inverse of the matrix with the numerical example and said, “Is it true? wrong?” he wanted to make an inference. However, this process performed by the pre-service teacher has nothing to do with the desired process pattern.

Table 4. The answers of the pre-service teachers about Question 2

Category	Code	f	%
Incorrect Answer	Numerical example	K2, K5, K6, K7, K8, K10, K12, K19, K20, K25, K26	%41
	inability to find error	K13, K18, K21, K22	%15
	Making proof	K11, K14, K15	%12
	Looking for the error in the wrong place	K1, K3, K16	%12
	Directly using the feature	K23, K24	%8
	Validation work	K4, K9	%8
	Using a unit matrix (special case)	K17	%4

When Table 4 is examined, it is seen that none of the pre-service teachers could form a correct or partially correct answer to the question, and all pre-service teachers gave incorrect answers to the question. While producing the wrong answers, eleven of the pre-service teachers were given the numerical example code, four of them were the code of not finding an error, three of them were the proof-making code, three of them were the code of searching the error in the wrong place, two of them were using the code directly, two of them were the validation study code, one of them was using the unit matrix (special case) code. Below are the visuals and comments of some pre-service teachers' answers to question 2.

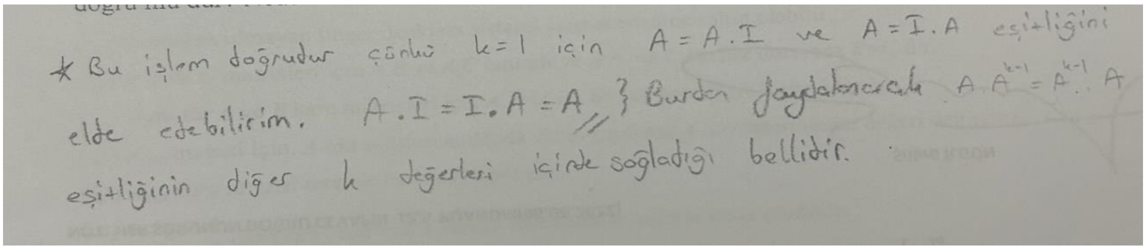

★ Bu işlem doğrudur çünkü $k=1$ için $A = A.I$ ve $A = I.A$ eşitliğini elde edebilirim. $A.I = I.A = A$ } Burdan faydalanarak $A.A^{k-1} = A^{k-1}.A$ eşitliğinin diğer k değerleri içinde sağladığı bellidir.

Figure 4. Example of the wrong answer given by the pre-service teacher to the question 2.

In Figure 4, the wrong answer given by the K17 pre-service teacher for the second question is seen. The pre-service teacher tries to show how to operate in exponentiation operations of matrices with a numerical assumption. While performing the operation, the pre-service teacher tried to prove the accuracy of the operation by giving a numerical value and as a result, he said that the unit matrix did not contribute to the operation, and he was convinced of the correctness of this force taking in the matrices. However, if he takes other values instead of k=1, he will realize that his acceptance is wrong.

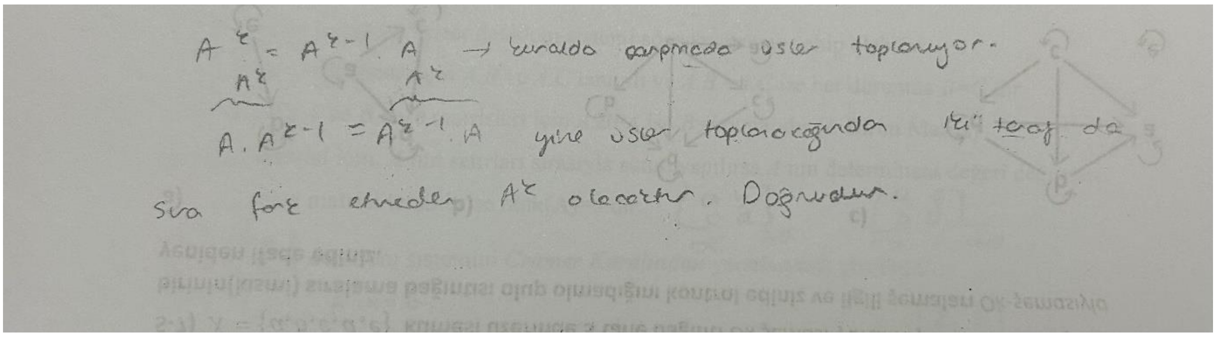

$A^k = A^{k-1}.A$ → burada çarpmada üsler toplanıyor.
$\underbrace{A.A^{k-1}}_{A^k} = \underbrace{A^{k-1}.A}_{A^k}$ yine üsler toplanacağından iki taraf da sıra fark etmeden A^k olacaktır. Doğrudur.

Figure 5. Example of the wrong answer given by the pre-service teacher to the question 2.

In Figure 5, the wrong answer given by the K1 pre-service teacher for the second question is seen. The pre-service teacher performs algebraic verification in a simple sense, considering the rule that the exponents are added while multiplying the base in exponential numbers. However, it is difficult to say that this rule, which is used in matrices and exponential numbers, is also valid in matrices. This shows that the pre-service teacher could not reach a certain level of proficiency in content knowledge and could not discover the main lines of the transferred subject. Again, it can be claimed that the pre-service teacher could not make sense of the operation of the combination and change features given in the matrix subject.

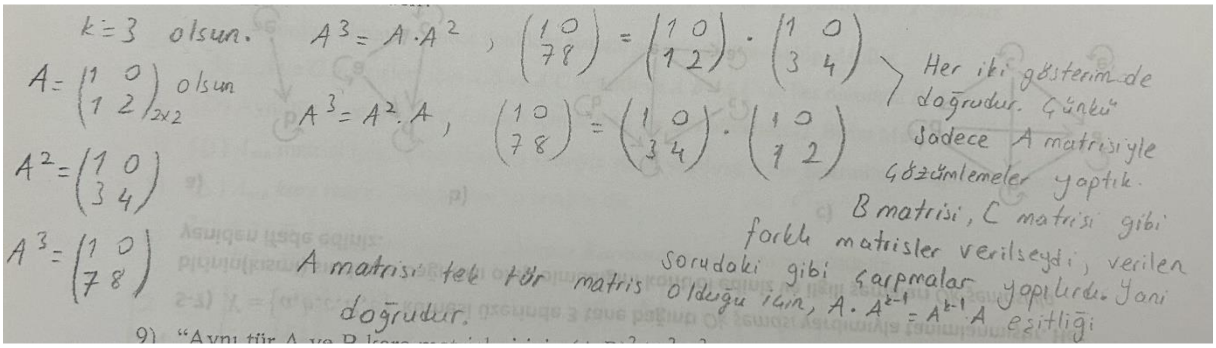


Figure 6. Example of the wrong answer given by the pre-service teacher to the question 2.

In Figure 6, the wrong answer given by K6 pre-service teacher for the 2nd question is seen. Generally, the correctness of the process can be determined by taking the numerical value larger in verification processes. Here, too, the pre-service teacher wants to show the accuracy of the feature given by the shortcut or making exponentiation by taking k=3. The pre-service teacher sees that this assumption is also correct. However, the pre-service teacher could not realize that the commutative feature of matrices was accepted under certain conditions and could not realize that the commutative feature of what was given here should be made within the unification feature.

Table 5. The answers of the pre-service teachers about Question 3

Category	Code	f	%
Correct Answer	Demonstration with proof	K1, K14, K19	%12
	Directly using the feature	K20	%4
Partially Correct Answer	Giving example to the contrary	K11, K12, K21, K22, K23, K25	%23
	Demonstration with proof	K3, K6, K24	%11
	Numerical example	K5	%4
Incorrect Answer	Trying to persuade	K2, K7, K8, K13, K18	%19
	Demonstration with proof	K9, K16, K17	%11
	Looking for the error in the wrong place	K4, K15	%8
	No explanation	K10	%4
No Answer	Empty	K26	%4

Looking at Table 5, it is seen that the pre-service teachers produced answers to the questions in all four categories. Four of the pre-service teachers produced correct answers to the given question and made the correctness of the questions by referring to the code 'demonstration with proof, directly using the feature', ten of the pre-service teachers produced a partially correct answer to the given question and benefited from the code 'contrary example, demonstration with proof, giving numerical example', and it is seen that ten pre-service teachers are in the wrong answer category and they used the code 'trying to persuade, looking for the error in the wrong place' while making this wrong answer category. Below are the visuals and comments of some pre-service teachers' answers to question 3.

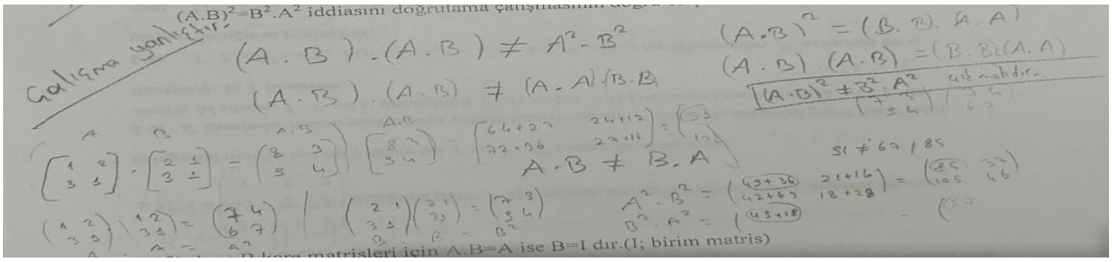


Figure 7. Example of the correct answer given by the pre-service teacher to the question 3

Figure 7 shows the correct answer given by the K14 pre-service teacher for the 3rd question. While writing the square of (AB), the pre-service teacher thinks that the writing will be in the form of (AB). (AB) with the logic of exponential expression in simple terms. He then supports this assumption with a numerical example and presents the procedural (operations) structures to ensure equality. In addition, the pre-service teacher demonstrates the correctness of his claim by referring to the property of change. In other words, revealing the existence of the commutative property is the basic structure for the correctness of what is claimed.

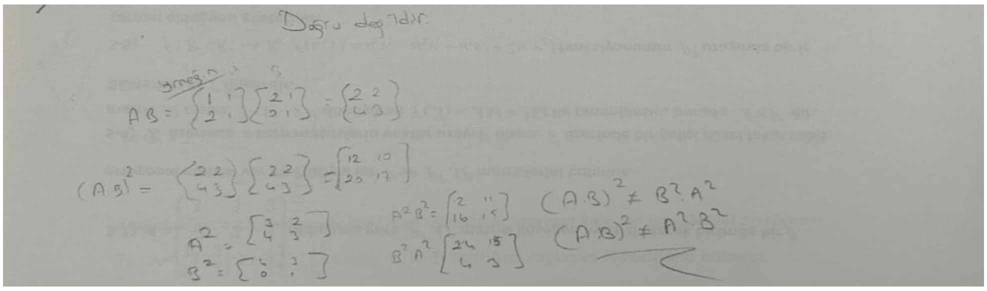
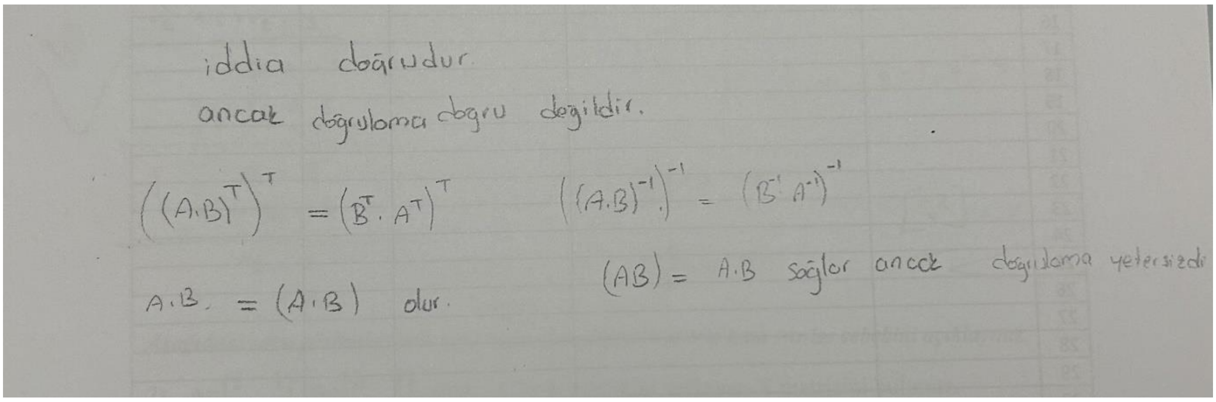


Figure 8. Example of partially correct answer given by the student to the question 3.

Figure 8 shows the partially correct answer given by the K12 pre-service teacher for the 3rd question. The pre-service teacher starts the process with the right step. In other words, he sees the result by writing the expression (AB) and then tries to say the existence of what is claimed by finding the square of (AB). The fact that the result he found did not support the alleged one might have convinced the pre-service teacher that his representation was correct. However, the fact that the intellectual structure depending on the change feature could not be said could not lead the pre-service teacher to the right.



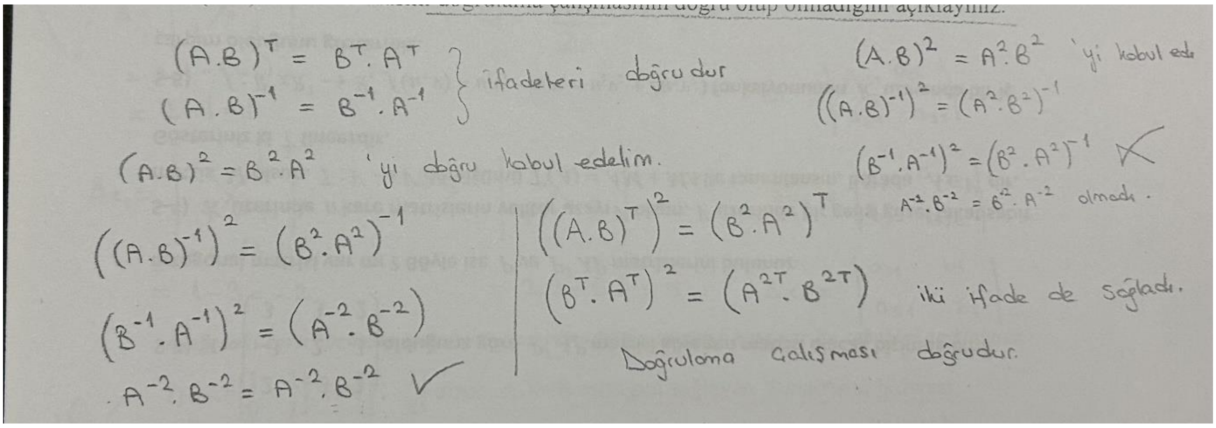
iddia doğrudur
ancak doğrulama doğru değildir.

$((A.B)^T)^T = (B^T.A^T)^T$ $\quad ((A.B)^{-1})^{-1} = (B^{-1}A^{-1})^{-1}$

$A.B = (A.B)$ dur. $\quad (AB) = A.B$ sağlar ancak doğrulama yetersizdi

Figure 9. Example of wrong answer given by the pre-service teacher to the question 3

In Figure 9, the wrong answer given by the K16 pre-service teacher for the 3rd question is seen. The pre-service teacher tries to explain the verification process by referring to the premises in the question. But the premises given are already theoretical assumptions about whether the claim is true or not. In other words, it is among the expected answers to see whether the theoretical assumptions work for accuracy. Not being able to notice the change feature leads the pre-service teacher to wrong.



$(A.B)^T = B^T.A^T$
$(A.B)^{-1} = B^{-1}.A^{-1}$ ifadeleri doğrudur

$(A.B)^2 = B^2.A^2$ 'yi doğru kabul edelim.

$((A.B)^{-1})^2 = (B^2.A^2)^{-1}$

$(B^{-1}.A^{-1})^2 = (A^{-2}.B^{-2})$

$A^{-2}.B^{-2} = A^{-2}.B^{-2}$ ✓

$((A.B)^T)^2 = (B^2.A^2)^T$

$(B^T.A^T)^2 = (A^{2T}.B^{2T})$ iki ifade de sağladı.

Doğrulama çalışması doğrudur.

$(A.B)^2 = A^2.B^2$ 'yi kabul ed

$((A.B)^{-1})^2 = (A^2.B^2)^{-1}$

$(B^{-1}.A^{-1})^2 = (B^2.A^2)^{-1}$ ✗

$A^{-2}.B^{-2} = B^{-2}.A^{-2}$ olmadı.

Figure 10. Example of the wrong answer given by the pre-service teacher to the question 3

In Figure 10, the wrong answer given by the K9 pre-service teacher for the 3rd question is seen. When the answer is examined, it shows that the pre-service teacher is sure that the given premises are correct. Accepting the ergi method to be is another right of the pre-service teacher. However, the most important reason for the pre-service teacher to go wrong is the incorrect expression of the replacement procedure in the powers of exponential expressions. In other words, the operations made for the power of real numbers have been tried to be done for the matrix. However, this is not possible in matrices. This made the pre-service teacher go wrong.

Table 5. Students' answers to Question 4

Category	Code	F	%
Correct Answer	Giving example to the contrary	K14, K19, K20	%12
Partially Correct Answer	Validation work	K23	%4
Incorrect Answer	No explanation	K15, K16, K18, K21, K24, K25, K24	%27
	Checking the solution by repeating it	K2, K3, K4, K6, K9	%19
	Looking for the error in the wrong place	K1, K7, K11, K12	%15
	Directly using the feature	K8, K13, K17, K21	%15
	Numerical example	K5, K10	%8

When Table 5 is considered, it is seen that three of the pre-service teachers produced correct answer, one partially correct answer, and twenty-one incorrect answers. It is seen that the pre-service teachers use the code 'giving example to the contrary' for the correct answer, the 'verification work' code for the partial verification, and the codes of 'checking the solution by repeating the exact same answer, looking for the error in the wrong place, directly using the feature, giving a numerical example' for the wrong answer. Below are the visuals and comments of some pre-service teachers' answers to question 5.

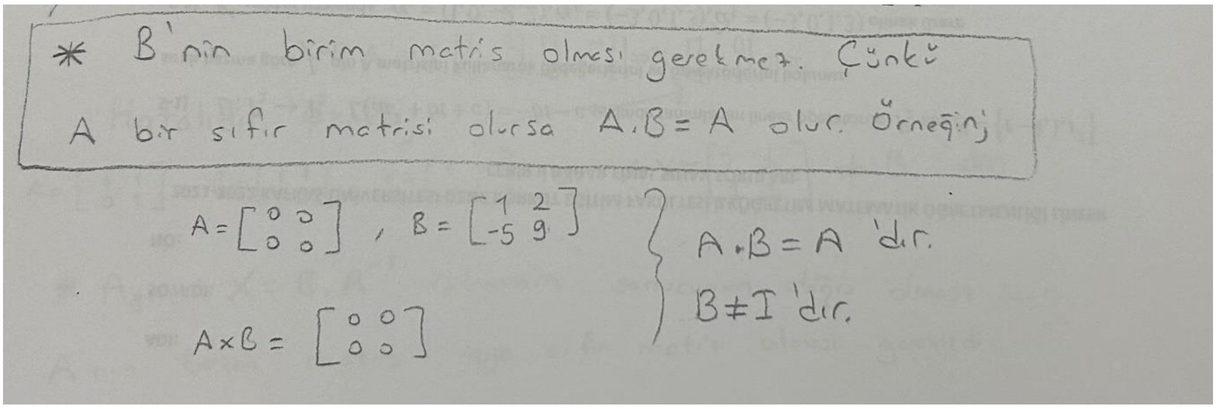
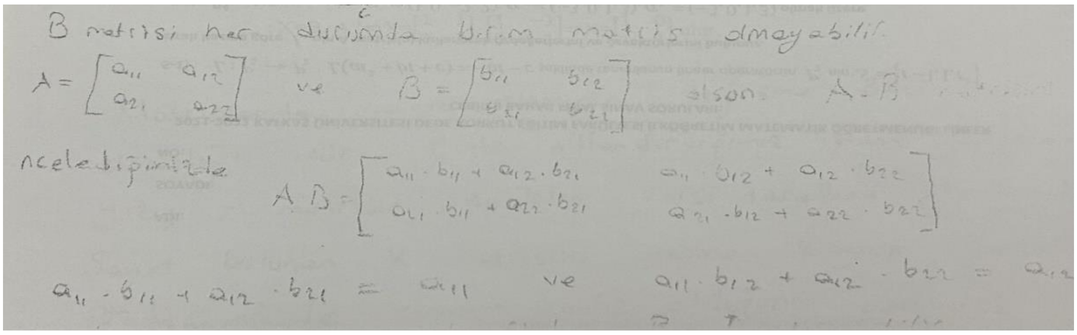


Figure 11. Example of the correct answer given by the pre-service teacher to the question 4

Figure 11 shows the correct answer given by the K19 pre-service teacher for the 4th question. While the pre-service teacher was expressing the correct answer, he started with a correct acceptance. In fact, it can be said that he started with a correct sampling. For the correctness of the claim, the pre-service teacher's handling of the zero matrix, revealing the falsity of the proven equality in a short way, convinced the candidate that he found the truth. Thus, he realized that the B matrix would not be the unit matrix and formed the correct answer.


B matrisi her durumda birim matris olmayabilir.

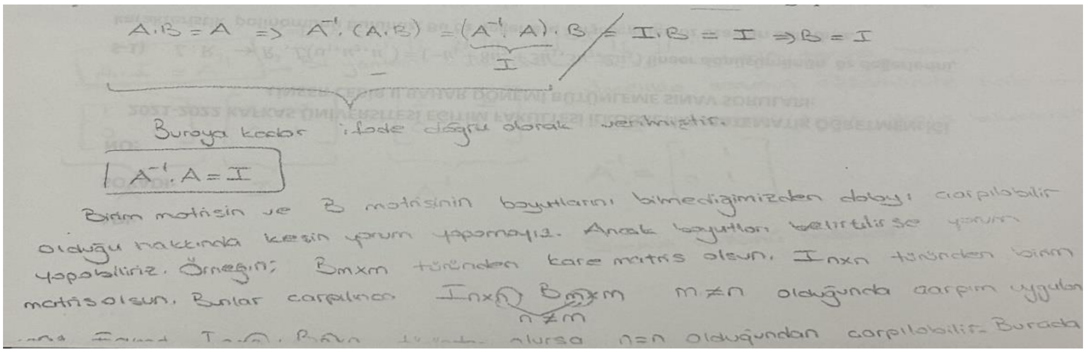
$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ ve $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$ olsun. $A.B$

incelediğimizde $A.B = \begin{bmatrix} a_{11}.b_{11} + a_{12}.b_{21} & a_{11}.b_{12} + a_{12}.b_{22} \\ a_{21}.b_{11} + a_{22}.b_{21} & a_{21}.b_{12} + a_{22}.b_{22} \end{bmatrix}$

$a_{11}.b_{11} + a_{12}.b_{21} = a_{11}$ ve $a_{11}.b_{12} + a_{12}.b_{22} = a_{12}$

Figure 12. Example of partially correct answer given by the student to the question 4

In Figure 12, the partially correct answer given by the K23 pre-service teacher for the 4th question is seen. The fact that the B matrix in the beginning will not be a unit indicates that the pre-service teacher is correct. The pre-service teacher considers the correctness of the proven equality by taking any two matrices. However, the inability to terminate these correct processing processes could not lead the pre-service teacher to a clear correct answer.


$\underbrace{A.B = A \Rightarrow A^{-1}.(A.B) = (\underbrace{A^{-1}.A}_{I}).B = I.B = I \Rightarrow B = I}$

Buraya kadar ifade doğru olarak verilmiştir.

$\boxed{A^{-1}.A = I}$

Birim matrisin ve B matrisinin boyutlarını bilmediğimizden dolayı çarpılabilir olduğu hakkında kesin yorum yapamayız. Ancak boyutları belirtilirse yorum yapabiliriz. Örneğin; $B_{m \times m}$ türünden kare matris olsun. $I_{n \times n}$ türünden birim matris olsun. Bunlar çarpılınca $I_{n \times n} B_{m \times m}$ $m \neq n$ olduğunda çarpım uygulan-

$n \neq m$

... olursa $n = n$ olduğundan çarpılabilir. Burada

Figure 13. An example of an incorrect answer given by the student to the question 4

In Figure 13, the wrong answer given by the K7 pre-service teacher for the 4th question is seen. At the beginning, the pre-service teacher starts the process with correct reasoning and knowledge. Then, the inference about the type of matrices misleads the pre-service teacher. Because the claimed equality representation is not related to the type (order) of the matrices. The pre-service teacher who correctly established the relation of matrix order with multiplication in matrices misinterpreted the acceptance.

Discussion and Conclusion

Considering the findings as a result of the study, first of all, the dimension of the pre-service teachers' perspectives on the questions was discussed. Within the framework of this dimension, it was discussed what the pre-service teachers thought during the answering process, how they transferred their knowledge to the solutions, how much they benefited from the teaching approaches presented to them in their classes, and as a result of all these, what errors were made by the pre-service teachers in the solutions. Because all the points mentioned are important clues that show the pre-service teachers' own approaches, their knowledge and how they transfer this knowledge to papers. When we look at the answers given to the questions in general, it is seen that the pre-service teachers are relatively successful in saying the existence of an error in a linear equation question by finding the inverse matrix in the matrices. It is noticed that the pre-service teachers have no learning at all in telling how the exponentiation affects the change and unification properties of the matrices. It can be shown that the most partially correct answer is the third question, and the reason for this is that pre-service teachers test their fundamental exponentiation skills and their operational skills in the form of searching for the truth by giving numerical examples. Their moderate performance may be due to the fact that linear algebra consists of a long list of rules, as stated in Skemp's theory of comprehension, and that relational understanding is not given importance.

When we analyze the questions asked within the scope of the research one by one and examine how the pre-service teachers produce answers to the questions; ,in the first question given to the pre-service teachers, 'Gauss-Jordan reduction method, Cramer Method, Solution with the Inverse of Coefficients Matrix' types are used in the solution of linear equation systems. This question is about testing how well the solution logic with the Inverse of the Coefficients Matrix, which is one of the solution types of linear equation systems on matrices, is met by the pre-service teachers. *A and B* matrices are given in the question and the accuracy of the $X = BA^{-1}$ operation is questioned. When the answers of the candidates were examined, it was tried to show how to produce an answer using A^{-1}. It was seen that most of the candidates who followed this path were successful, but many of them could not reach the answer because they incorrectly determined the use of A^{-1} in the procedure. In particular, it has been noticed that the pre-service teachers make a lot of reference to the feature of change in matrices within the scope of this question and they misunderstand its meaning. It has been determined that most of the pre-service teachers have the necessary conceptual knowledge and procedural functioning for the question, but they have difficulties in using this knowledge and executing the procedures. These results are compatible with Dorier and Sierpinska's (2001) idea that linear algebra is a cognitively and conceptually difficult subject, and that conceptual difficulties and cognitive difficulties hinder pre-service teachers' learning.

In the second question given to the pre-service teachers, the expressions $A^k = A.A^{k-1}$ and $A^k = A^{k-1}. A$ were given for $k \in^+$, which is related to the power of the matrices, and the accuracy of the equation was tested by reminding that the multiplication operation in matrices does not have a commutative property. When the answers of the pre-service teachers are examined, it is seen that they generally use the known exponential number rule in real numbers. In other words, they said that the given statement is true by considering the principle of adding the exponents if the bases are equal in exponential numbers. Again, the pre-service teachers tried to achieve equality by giving numerical values. As a different point of view, some of the pre-service teachers showed that both expressions are equal by adding the inverse of the matrix A to both sides of the equation in operational terms. Considering all these, it is seen that the pre-service teachers have theoretical knowledge about the given process in their minds, but they do not have any comments on how to operate this theoretical knowledge in matrices. We can think that the reason for this situation is the understanding that the mathematics principles that the pre-service teachers learn are applied in a similar way in all subjects of mathematics. The main reason for the operational errors made in the matrices may be the misconceptions of the students in this subject (Hidayanti, 2020).

In the third question, starting from the premise that the square of the product of two matrices is not equal to the product of the squares of the individual matrices ($(A.B)^2 \neq A^2B^2$), by giving the premise that the transpose of the product of the matrices is the transpose of the individual matrices ($(A.B)^T = B^T A^T$), the inverse of the multiplication of the matrices is the multiplication of the inverse of the individual matrices ($(A.B)^{-1} = B^{-1}A^{-1}$), verifying the equation of $(A.B)^2 = B^2A^2$ is tried to performed. When the answers of the candidates were examined, it was determined that the pre-service teachers did not understand why the premises were given and what they would do. However, it was observed that some of the pre-service teachers made operations in the form of $((A.B)^{-1})^2 = (B^2A^2)^{-1}$ and that the pre-service teachers were of the opinion that the power change of exponential expressions in real numbers can be made in matrices as well as in matrices. It is also seen that the commutative property is used again for this question. Within the scope of the research, it is striking that only one pre-service teacher did not answer this question, in which the pre-service teachers generally answered the questions. All these show that pre-service teachers do not understand the proof approach and do not know the proof methods very well. In special cases, it is seen that students cannot produce solutions and they fall into misconceptions in the solutions they produce (Mulyatna & Nurramah, 2020).

In the fourth question, if $A.B = A$ is, then is $B = I$ possible? The existence of such an approach is discussed. The aim here is to make the pre-service teachers fall into the misconception that the matrix is really equal to the unit matrix by operating the inverse of the matrix on both sides of the equation. When the answers of most of the pre-service teachers were examined, it was seen that they acted in this way and fell into the predicted error. In other words, pre-service teachers did not realize the error. Again, many pre-service teachers tried to show the correctness of the equality by giving numerical values. This shows that the pre-service teachers still maintain the habit of finding the answer by giving the value they acquired in secondary education. In this case, it can be said that wrong learning is not abandoned. Some studies, similar to this result, have shown that; Although students can perform operations that require calculation in Linear Algebra, they have difficulties in understanding concepts and establishing relationships between concepts (Dorier, 1998; Harel, 1989/b).

Considering the concepts in Linear Algebra and operations with these concepts, it is possible to say that it can be expressed as a combination of different disciplines and fields in mathematics. The contribution of Linear Algebra (Mostow and Sampson, 1969), not only in mathematics itself, but also in the theoretical and practical development of other branches of science (Carlson, 1993; Çallıalp, 1994; Kuiper, 1963; Roman, 1984), Linear Algebra in and outside of mathematics, has made it one of the most useful theories (Harel, 1987, 1989/a, 1989/b; Strang, 1988). Therefore, the fact that Linear Algebra should be found in all areas of life as well as in mathematics itself (Harel, 1989/a) reveals that it is necessary to focus on teaching linear algebra. Park City Mathematics Institute stated that the field of learning and teaching that left the biggest impact on them among all the fields they did was Linear Algebra. They explained that they think this situation raises doubts about how students learn in the field of linear algebra. Studies in this area indicate that learning and teaching linear algebra is difficult (Hillel & Sierpinska, 1993; Dorier & Sierpinska, 2001).

In general, it is seen that pre-service teachers make errors and misconceptions due to procedural structures, conceptual errors and lack of conceptual knowledge (Ndloyu, 2019; Mutambara & Bansilal, 2022). In particular, when we look at the literature, researches have been made on systems of equations in linear algebra, matrices and operations on matrices, linear independence-dependence, vector space, base and dimension. The results found are that students have difficulties and make errors in these subjects. The students who took this course complained about not being able to connect the concepts of the Linear Algebra course with the other subjects of mathematics they had learned before, not being able to learn these concepts concretely because they could not perceive them concretely, and having to learn many concepts, one after the other, that they had never heard of before (Dorier, 2002). Similar results were found with the error-based activities performed in this study. Linear Algebra course requires high-level mental skills such as analyzing, making assumptions, testing, and generalizing in terms of its general abstract structure, learning its concepts, and understanding the relations between these concepts. Organizing the teaching of linear algebra course to cover various understanding dimensions to students can increase performance and retention. For this purpose, by using error-based activities in the teaching of the linear algebra course, it can enable them to reach the correct results with the errors given in the questions, and make the expression more concrete and understandable.

Policy Implications

Many mathematical mistakes made today and how these mistakes are made show us both the procedural and conceptual knowledge of the students. procedural and conceptual knowledge are not independent of each other. These information reinforce each other. Therefore, a balance should be established between both procedural and conceptual knowledge in teaching. In general, students memorize information about each concept instead of learning basic information in mathematics teaching. This is also the case in higher education. In the courses given in undergraduate education, students use their procedural knowledge. but conceptual information is not used. The properties of mathematical rules, principles or concepts in conceptual knowledge and associating these concepts with each other are important for both the teacher and the student.

Conflict of Interest

The corresponding author states that there is no conflict of interest on behalf of all authors.

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Ethical Statement

We act in accordance with scientific ethical principles and rules from all stages of the study, including preparation, data collection, analysis and presentation of information; we have cited all data and information not obtained within the scope of this study and we have included these sources in the bibliography; we declare that we have not made any changes in the data used and that we comply with ethical duties and responsibilities by accepting all the terms and conditions of the Committee on Publication Ethics (COPE). At any time, we declare that we consent to all moral and legal consequences that may arise in the event that a situation contrary to this statement we have made regarding the study is detected.

References

Abidin, S. N. Z., Zain, S. M. M., Hamzah, H. H. M., & Abd Rahim, N. Z. (2016). Mısconceptıons In Usıng Gauss-Jordan Elımınatıon Method By Undergraduate Students. In *6th International Conference on Global Social Entrepreneurship (Kota Bharu)* (p. 291)

Alpaca, N., Aparicio, V., Sharhorodska, O. & Jaen, D. (2020). Problem based Learning with Information and Communications Technology Support: An Experience in the Teaching-Learning of Matrix Algebra. International Journal of Advanced Computer Science and Applications,(11)3.https://pdfs.semanticscholar.org/1156/a172278afe5b719ffc98009e17780225f2fa.pdf

Altıntaş, E., İlgün, Ş., & Daştan, Z. (2020). Reflections from 7th Grade Students Compared to Error Based Questions in Mathematics. In K. A. KIRKIC, Innovative Learning and Teaching Practices (p. 245-279). Ankara: İksad Publishing.

Altıntaş, E., İlgün, Ş., & Daştan, Z. (2020). Examining 7th Grade Students with Questions Containing Wrong Solutions in Mathematics. Turkish Studies, 15(6), 3939-3957. DOI: 10.47423/TurkishStudies.46584

Altıntaş, E., İlgün, Ş., Uygun, S., & Angay, M. (2021). Errors and Misconceptions of 7th Grade Students About First-Order Equations With One Unknown. E-Kafkas Journal of Educational Research, 8(3), 788-820. DOI:10.30900/kafkasegt. 1003487

Arnawa, I. M., & Nita, S. (2020). Improvement students' achievement in elementary linear algebra through APOS theory approach. Journal of Physics: Conference Series, 1567(2), p.022080. IOP Publishing.

Arnawa, I., Ginting, B., & Nita, S. (2021). Does the Use of APOS Theory Promote Students' Achievement in Elementary Linear Algebra?. International Journal of Instruction, 14(3), 175-186. https://doi.org/10.29333/iji.2021.14310a

Arnawa, Nita, Yerizon .(2019). Errors and misconceptions in learning elementary linear algebra. Journal of Physics: Conference Series (JPCS), 1321. DOI: 10.1088/1742-6596/1321/2/022095

Arslan, K. (2021). Investigation of elementary mathematics teaching program students' ability to perceive linear dependency-linear independence and base-dimension concepts in vector spaces (Master's thesis, Graduate School of Sciences/Elementary Mathematics Teaching Department).

Aygor, N. & Ozdag, H. (2012). Misconceptions in linear algebra: the case of undergraduate students. Procedia-Social and Behavioral Sciences, 46, 2989-2994. https://doi.org/10.1016/j.sbspro.2012.05.602

Bağdat, O. & Koparan, M. (2017). Errors of Secondary School Mathematics Teacher Candidates in Problems Using Primitive Line Operations in Linear Algebra Lesson. Route Educational and Social Science Journal, 4(2).

Baştürk, S. (2014). The place of student error in mathematics teaching: The concept of error and obstacle. Education in the Light of Science and Reason, 166, 14-23. https://www.researchgate.net/publication/312295181

Bilgili, S., Özkaya, M., Çiltaş, A., & Konyalıoğlu, A. C. (2020). Examination of middle school mathematics teachers' error approaches to modelling. Anemon Mus Alparslan University Journal of Social Sciences, 8(3), 871-882. DOI: 10.7822/omuefd.845782

Birinci, D., Delice, A. & Aydın, E. (2014). University students' solution processes in systems of linear equation. Procedia-Social and Behavioral Sciences, 153, 563-568. http://dx.doi.org/10.1016/j.sbspro.2014.09.244

Büyüköztürk, Ş., Kılıç Çakmak, E., Akgün, Ö. E., Karadeniz, Ş. ve Demirel, F. (2018). Scientific Research Methods. (24th Edition). Ankara: Pegem Academy Publications.

Gökçek, T. & Davey, L. (2009). Application of Case Study Assessments. Primary education Online , 8 (2) , 1-3 . Retrieved from https://dergipark.org.tr/en/pub/ilkonline/issue/8598/107042

Demirci, Ö., Özkaya, M., & Konyalıoğlu, A. (2017). Error approaches of prospective teachers about probability. Journal of Erzincan University Faculty of Education, 19(2), 153-172. https://doi.org/10.17556/erziefd.310667

DeVries, D., & Arnon, I. (2004). Solution--What Does It Mean? Helping Linear Algebra Students Develop the Concept While Improving Research Tools. International Group for the Psychology of Mathematics Education, Vol 2, 55-62. https://files.eric.ed.gov/fulltext/ED489680.pdf

Dorier, J. L., & Sierpinska, A. (2001). Research into the teaching and learning of linear algebra. The teaching and learning of mathematics at university level (pp. 255-273). Springer, Dordrecht.

Durkaya, M., Aksu, Z., Öçal, M. F., Şenel, E. Ö., Konyalıoğlu, A. C., Hızarcı, S., & Kaplan, A. (2011). Secondary school mathematics teachers' approaches to students' possible mistakes. Procedia-Social and Behavioral Sciences, 15, 2569-2573. http://dx.doi.org/10.1016/j.sbspro.2011.04.147

Durkaya, M., Şenel, E. Ö., Öçal, M. F., Kaplan, A., Aksu, Z., & Konyalıoğlu, A. C. (2011). Pre-service mathematics teachers' multiple representation competencies about determinant concept. Procedia-Social and Behavioral Sciences, 15, 2554-2558. http://dx.doi.org/10.1016/j.sbspro.2011.04.144

Erdem, Z. Ç., & Gürbüz, R. (2017). An investigation on students' mistakes and misconceptions: Example of equation. Van Yüzüncü Yıl University Journal of the Faculty of Education, 14(1), 640-670. http://dx.doi.org/10.23891/efdyyu.2017.25

Hidayanti, N. (2020). Analysis of Student Mistakes in Completing Matrix Problems in The Linear Algebra Course. *Journal of Physics: Conference Series (JPCS), 1477.* DOI**:**10.1088/1742-6596/1477/4/042046

Kardeş, D. (2010). Examination of mathematics teacher candidates' linear equation systems solution processes in the context of self-efficacy perception and multiple representations. (Doctoral dissertation, Marmara University, Türkiye).

Kazunga, C., & Bansilal, S. (2017). Zimbabwean in-service mathematics teachers' understanding of matrix operations. The Journal of Mathematical Behavior, 47, 81-95. http://dx.doi.org/10.1016/j.jmathb.2017.05.003

Konyalıoğlu, A. C., Özkaya, M., & Altun Gedik, S. D. (2012). Investigation of Pre-Service Mathematics Teachers' SubjectMatter Knowledge in terms of Their Approaches to Errors. Iğdır University Journal of the Institute of Science and Technology, 2(2), 27-32.

Konyalioglu, A. C., Aksu, Z., Senel, E. O., & Tortumlu, N. (2010). The investigation of mathematics teachers cadidates' skills of interrogating the reasons of errors in the process of solving mathematics questions. Proceedings of the International Symposium on Teacher Education Policies and Issues II, 951-960.

Konyalioglu, A. C., İpek, A. S., & Isik, A. (2003). On the teaching linear algebra at the university level: the role of visualization in the teaching vector spaces. Research in Mathematical Education, 7(1), 59-67.

Konyalioglu, A. C., Konyaliglu, S., & Isik, A. (2008). Effectıveness Of Vısualızatıon Approach On Student's Conceptual Learnıng. Journal of Qafqaz University, (24).

Konyalioglu, S., Konyalioglu, A. C., Ipek, A. S., & Isik, A. (2005). The role of visualization approach on student's conceptual learning. International Journal for Mathematics Teaching and Learning, 47, 1-9.

Köğce, D. (2022). Examination of Mathematical Errors and Mistakes in Calculus Course. Educational Policy Analysis and Strategic Research, 17(1), 295-311. doi: 10.29329/epasr.2022.248.15

Msomi, A. M., & Bansilal, S. (2022). Analysis of Students' Errors and Misconceptions in Solving Linear Ordinary Differential Equations Using the Method of Laplace Transform. International Electronic Journal of Mathematics Education, 17(1), 670. https://doi.org/10.29333/iejme/11474

Mulyatna, F., & Nurrahmah, A. (2020). Look Same but Different: Misconceptions in Linear Combinations. In Semantık Conference of Mathematics Education (SEMANTIK 2019), 1-5, Atlantis Press.

Mutambara, L. H. N., & Bansilal, S. (2022). Misconceptions and resulting errors displayed by in service teachers in the learning of linear independence. International Electronic Journal of Mathematics Education, 17(4), em0716. https://doi.org/10.29333/iejme/12483

Mutlu, Y., & Söylemez, İ. (2018). Examination of master's and doctoral theses on mathematical misconceptions. Başkent University Journal of Education, 5(2), 187-197.

Mutuk, H. (2018). Matrix Algebra with Mathematica. Karadeniz Journal of Science, 8(1), 115-130. https://doi.org/10.31466/kfbd.420864

Ndlovu, Z. (2019). An analysis of pre-service mathematics mistakes in matrices. International Journal, 75(3/1). http://dx.doi.org/10.21506/j.ponte.2019.3.9

Önal, H. & Aydın, O. (2018). Misconceptions and error examples in primary school mathematics lesson, Journal of Educational Theory and Practice, 4(2), 1-9. https://dergipark.org.tr/tr/pub/ekuad/issue/38280/442698

Önmez, T. (2015). A qualitative study on elementary mathematics teacher candidates' understanding of linear equations (Master's thesis, Necmettin Erbakan Üniversitesi).

Principles, N. (2000). Standards for school mathematics Reston, VA National Council Teaching Mathematics.

Turgut, M. (2010). The Effect of Technology Supported Linear Algebra Teaching on Primary Mathematics Teacher Candidates' Spatial Abilities. (Master's thesis, Dokuz Eylül University Institute of Educational Sciences, İzmir.)

Yenilmez, K., & Yaşa, E. (2008). Elementary school students' misconceptions in geometry. Journal of Uludag University Faculty of Education, 21(2), 461-483. https://dergipark.org.tr/tr/download/article-file/153350

Yıldırım, A. ve Şimşek, H. (2018). Qualitative research methods in the social sciences. (11th Edition). Ankara: Seçkin Publishing.

Zembat, İ, Ö. (2010). What is misconception?. MF. Özmantar, E. Bingölbali and H. Akkoç (Edıtors), Mathematical Misconceptions and Solution Suggestions. Ankara : Pegem A publishing.